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ORIGINAL PAPER

Altruistic giving and risk taking in human affairs

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Abstract

The purpose of this paper is to provide a general proposition of the relationship between altruism and risk taking. As explained in the body of the paper, we diverge from a result reported in Stark et al. (2022) and provide an expansion and a generalization of a preliminary result reported in Stark (2024). In a broad utility framework, we study the risk aversion of an altruistic person who is an active donor (benefactor) and the risk aversion of a beneficiary of an altruistic transfer. In both cases, we find that altruism lowers risk aversion. The specific case in which the utility functions of the benefactor and of the beneficiary are constant relative risk aversion (CRRA) functions constitutes a vivid example of lesser risk aversion characterization. We conclude that in terms of risk-taking behavior, a "population" endowed with altruism is uniformly more willing to take risks than a comparable "population" devoid of altruism.

Keywords: Altruism; Altruistic transfers; The absolute risk aversion of a practicing altruistic person; Intensity of altruism; Variation in risk-taking preferences; The absolute risk aversion of a beneficiary of an altruistic transfer

JEL classification: D01; D64; D81; G41

1. Introduction

Assuming that in the formation of risk-taking preferences, relating to others matters, we study the case in which a person relates to others altruistically. The need to

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conduct such an inquiry arises not merely because altruism is common and plays an important role in the affairs of individuals, families, and groups of various types, but also because it is unclear in what way altruism will influence the likelihood of an altruistic person to take risks. Will this person's risk-taking behavior be different if the utility of another person does not enter his utility function?

There are several reasons why it is important to study the effect of altruism on risk aversion and the propensity to take risks. First, the large literature on altruism has evolved independently of the large literature on risk taking. For example, while in the 2006 two-volume set *Handbook of the Economics of Giving, Altruism and Reciprocity* (Kolm and Mercier Ythier, 2006) altruism as a trait is referred to thousands of times, the risk aversion trait of an altruistic person appears nowhere. Second, there is a general presumption that encouraging and promoting altruistic behavior is socially desirable, and that lower risk aversion encourages people to pursue risky ventures which could contribute to innovation, economic growth, and social welfare. Thus, establishing the connection between the two is valuable. Third, suppose that it is found that altruism causes people to be more willing to take risks, and that, for the reasons alluded to above, there is a social preference to induce people to become less reluctant to resort to risk-taking behavior. Then, instilling altruistic proclivities becomes an effective intervening policy tool. Fourth, abstract reasoning alone cannot determine the nature of the association: does being altruistic cause a person to become more reluctant to take risks because a risky undertaking turning sour will also damage his ability to make altruistic transfers? Or does altruism induce a person to resort to risky behavior because the reward for a successful outcome is amplified by the outcome facilitating a bigger transfer to the beneficiary of the altruistic transfer? Thus, rigorous analysis is required. Fifth, there is the issue of an incomplete mapping of preferences for risk taking. Several recent papers - including Stark and Zawojska (2015), Stark and Szczygielski (2019), Stark et al. (2019), and Stark (2020) - show that in the formation of risk-taking preferences, relating to others matters, and that subject to the modeling used, it is possible to identify the effect of that relationship on attitudes towards risk taking. In the models developed in these recent papers, the utility of the reference person is expanded into an additively separable function, where the added "social ties" component is accorded a weight that reflects its importance. And this component enters the function negatively: low relative wealth, low rank, and low status affect wellbeing adversely. Missing from these inquiries is a study of the case in which a person relates to others positively, namely altruistically. This is the inquiry that we undertake in this paper.

There is an obvious presumption that the beneficiary of altruistic transfers will be less averse to risks because the altruistic channel operates like an insurance arrangement. This response and the associated moral hazard were studied a long time ago (Bernheim and Stark, 1988). We address this issue in Claim 2. However, the attitude towards risk taking by the altruistic person requires close scrutiny. Holding other variables constant, is an altruistic person more risk averse or less risk averse than a comparable person who is not altruistic? Is the risk aversion of an altruistic person lower when the intensity of his altruism is higher? In the next section, we respond to these questions.

To the best of our knowledge, texts on altruism, spanning from the collection of studies in Phelps (1975) to Bourlès et al. (2021), have not addressed these two questions. When altruism and risk-taking behavior were linked, the context was the perception of the recipients of altruistic transfers that altruism provides them with a form of insurance. Recently, we made an effort to fill the research gap. As explained next, this effort met with only limited success.

In Stark et al. (2022), we made an initial attempt to forge a link between the trait of altruism and risk-taking preferences. The manner in which we formulated the research problem turned out to be wanting: we were unable to obtain results mathematically on the basis of a full derivative of the utility function of the altruistic person. The reason for this was that we could not accommodate the dependence of the altruistic transfer on the wealth of the altruistic person. To help resolve that difficulty, we outlined a justification for calculating the coefficient of the relative risk aversion of the altruistic person. We reported that we hold the level of the transfer as exogenous and that this treatment enables us to gauge the sensitivity of the coefficient of relative risk aversion to the intensity of the altruistic person's altruism. In our "defense," we remarked that if, alternatively, we were to calculate the coefficient of relative risk aversion as a full derivative of the utility function with respect to the level of wealth of the altruistic person, then the sensitivity of the coefficient of relative risk aversion with respect to the intensity of altruism would be nil because the coefficient will be a function of only the pretransfer levels of wealth of the altruistic person and the recipient of the altruistic transfer. As a consequence of pursuing that analytical approach, the results reported in Stark et al. (2022) were that an altruistic person is more risk averse than a nonaltruistic person and that the relative risk aversion of an altruistic person is higher when the intensity of his altruistic feelings is stronger. The realization that these results arose from a far too strict mathematical construct sent us back to the drawing board. The additional research effort has yielded two outcomes: a partial one, Stark (2024), and a comprehensive one, the current paper.

In Stark (2024), drawing on a logarithmic utility specification and assuming that an altruistic person engages optimally in a wealth transfer to the recipient of the altruistic transfer, we reported that the altruistic person is less risk averse than a person who is not altruistic. Aware that a more comprehensive analysis is warranted, we noted that to hand down a definitive verdict on whether altruism lowers risk aversion, it would not be enough to show that an altruistic person who is an active donor is less risk averse than a similar person who is not an active donor. It would also be necessary to show that the beneficiary of an altruistic transfer is less risk averse than a similar person who is not the beneficiary of an altruistic transfer. As already mentioned, there is an obvious presumption that the beneficiary of an altruistic transfer would be less averse to taking risks because the altruistic channel operates like an insurance arrangement. Still, although a presumption can guide formal inquiry, it cannot substitute for such inquiry. We also noted that the result reported in the 2024 paper was obtained on the basis of a logarithmic characterization of the altruistic person's utility and that this representation could be supplemented by the use of more general utility functions such as a constant relative risk aversion (CRRA) utility function.

This paper constitutes the comprehensive outcome. On the basis of a general specification of the research problem, we are able for the first time to draw a complete picture: we show that an altruistic person who is an active donor is *less* risk averse than a similar person who is not an active donor, that the higher the intensity of altruism of an altruistic person who is an active donor the *less risk averse* he is, and that under the condition that an altruistic person engages optimally in a wealth transfer to the recipient of an altruistic transfer, the latter person is *less risk averse* than he would have been had he not received a transfer. This rigorously derived set of results is novel.

The remainder of our paper is organized as follows. Our basic analytical framework and core results are presented in Section 2. In that section we provide the utility characterization, make and explain two technical assumptions, formulate two supporting lemmas, and then present our first claim, Claim 1, in which we characterize the risk-taking behavior of an altruistic person. We next formulate a third lemma and present our second claim, Claim 2, in which we characterize the risk-taking behavior of a recipient of an altruistic transfer. In Section 3, we show that the results delivered by the general model of Section 2 hold nicely in the special case of a CRRA utility function. Claims 3 and 4 in that section mirror, respectively, Claims 1 and 2 of Section 2. In brief, Section 4 demonstrates that Claims 3 and 4 hold true when, as in Bernheim and Stark (1988) and Stark (1999), the utility function is logarithmic. So as not to interrupt the flow of our main argument, we have relegated the proofs of our lemmas and claims, which at times are long and tedious, to the Appendices.

2. Characterizing the absolute risk aversion of an altruistic person, and the absolute risk aversion of a beneficiary of an altruistic transfer: A general formulation

Suppose that altruistic person i derives utility from his wealth, denoted by $w_i > 0$, and from the utility of person j. By $\alpha_i \in (0, 1)$ we denote the intensity of person i's altruism. The complementary weight, $1 - \alpha_i$, is accorded to the utility that person i obtains from his own wealth. Person i can transfer part of his wealth, t_i, to person j, where $0 \leq t_i < w_i$. By $w_j > 0$ we denote the pretransfer wealth of person j. The utility function of altruistic person i takes the form

$$u_i(w_i, w_j, t_i, \alpha_i) = (1 - \alpha_i)v(w_i - t_i) + \alpha_i u_j(w_j + t_i), \tag{1}$$

where $v(\cdot)$ is the utility that person i derives from his net wealth, and $u_j(\cdot)$ is the utility person j derives from his net wealth. Altruistic person i will transfer part of his wealth to person j as long as doing so will increase person i's utility. The optimal level of person i's utility is given by

$$u_i^*(w_i, w_j, \alpha_i) \equiv \max_{t_i \in [0, w_i)} u_i(w_i, w_j, t_i, \alpha_i). \tag{2}$$

We make the following three-part technical assumption regarding the functions $v(\cdot)$ and $u_j(\cdot)$.

Assumption 1. There exist $\underline{w} > 0$ and $\overline{w} > 0$, where $0 < \underline{w} < \overline{w}$, such that the following hold. (i) The functions $v(\cdot)$ and $u_j(\cdot)$ are strictly concave and three times continuously differentiable: $v(\cdot)$ on $(0, \overline{w})$, and $u_j(\cdot)$ on $(\underline{w}, \underline{w} + \overline{w})$. (ii) For every

$x_i \in (0, \overline{w})$ we have $v'(x_i) > 0$ and $v''(x_i) < 0$, and for every $x_j \in (\underline{w}, \underline{w} + \overline{w})$ we have $u_j'(x_j) > 0$ and $u_j''(x_j) < 0$. (iii) The right-hand derivative of $v(\cdot)$ approaches infinity at zero, that is, $\lim_{x_i \to 0^+} v'(x_i) = \infty$.

Given parts (i) and (iii) of Assumption 1, it follows that the solution of (2), that is, the optimal level of the wealth transfer, is unique for every $w_i \in (0, \overline{w})$ and $w_j \in (\underline{w}, \underline{w} + \overline{w})$. We denote this optimal transfer by $t_i^*(w_i, w_j, \alpha_i)$. For the sake of brevity, we subsequently drop the arguments w_i and w_j, and write $t_i^*(\alpha_i)$.

Following Pratt (1964) and Arrow (1965), the coefficient of absolute risk aversion (*ARA*) of person i is defined as

$$ARA_i \equiv -\frac{u_i''(w_i)}{u_i'(w_i)}.$$

In our setting,

$$ARA_i(\alpha_i) \equiv -\frac{\dfrac{\partial^2 u_i^*(w_i, w_j, \alpha_i)}{\partial w_i^2}}{\dfrac{\partial u_i^*(w_i, w_j, \alpha_i)}{\partial w_i}}.$$

Our interest is in ascertaining the relationship between absolute risk aversion and the intensity of altruism. Prior to formulating and proving our main results, we define four auxiliary functions:

$$A_i(x_i) \equiv -\frac{v''(x_i)}{v'(x_i)}, \quad A_j(x_j) \equiv -\frac{u_j''(x_j)}{u_j'(x_j)}, \quad B_i(x_i) \equiv \frac{1}{A_i(x_i)}, \quad \text{and } B_j(x_j) \equiv \frac{1}{A_j(x_j)},$$

where $x_i \in (0, \overline{w})$ and $x_j \in (\underline{w}, \underline{w} + \overline{w})$. From Assumption 1, it follows that $B_i(x_i) > 0$ and that $B_j(x_j) > 0$. Because by Assumption 1 the functions $v(\cdot)$ and $u_j(\cdot)$ are three times continuously differentiable on $(0, \overline{w})$ and on $(\underline{w}, \underline{w} + \overline{w})$, respectively, and because their first and second derivatives are, respectively, strictly positive and strictly negative, we see that $B_i(\cdot)$ and $B_j(\cdot)$ are continuously differentiable.

We also make a technical assumption regarding the levels of wealth of persons i and j.

Assumption 2. The level of wealth, $w_i \in (0, \overline{w})$, of person i and the level of wealth, $w_j \in (\underline{w}, \underline{w} + \overline{w})$, of person j satisfy the condition that for every $t_i \in [0, w_i)$,

$$B_i'(w_i - t_i) \le B_j'(w_j + t_i). \tag{3}$$

$B_i(x_i)$ is the inverse of the absolute risk aversion of altruistic person i, and $B_j(x_j)$ is the inverse of the absolute risk aversion of the recipient of altruistic transfer person j, henceforth the beneficiary. Therefore, $B_i'(w_i - t_i)$ is the post-transfer marginal value of the inverse of the altruistic person's absolute risk aversion, and $B_j'(w_j + t_i)$ is the post-transfer marginal value of the inverse of the beneficiary's absolute risk aversion. Condition (3) states that the marginal value of the inverse of the altruistic person's absolute risk aversion, taken following the transfer of wealth, is smaller than the marginal value of the inverse of the beneficiary's absolute risk aversion, taken following the receipt of wealth. The condition means that the sum $B_i(w_i - t_i) + B_j(w_j + t_i)$ (weakly) increases whenever the altruistic transfer increases.

We need Assumption 2 in order to subsequently prove that the absolute risk aversion of person i is decreasing with respect to the intensity of his altruism.[1] In Claim 1 (by means of Lemma 2), we show that Assumption 2 constitutes not only a sufficient condition but also a necessary condition for the absolute risk aversion of person i to decrease as the intensity of his altruism increases.

In addition, Assumption 2 will be of use when we set out to prove that altruistic person i is less risk averse than a comparable nonaltruistic person, and it will be drawn upon when we prove that the beneficiary of an altruistic transfer is less risk averse than a comparable person who is not a beneficiary of an altruistic transfer.

Drawing on Assumption 1, we next formulate and prove several lemmas. Inspired by Topkis (1978, 1998), we first introduce the following definition, terminology, and property.

Definition 1. We say that the function $u_i(w_i, w_j, t_i, \alpha_i)$ has increasing differences in (α_i, t_i) if for any $t_i^1 \in [0, w_i)$ and $t_i^2 \in [0, w_i)$ such that $t_i^2 > t_i^1$, the function

$$\alpha_i \in (0, 1) \to u_i\left(w_i, w_j, t_i^2, \alpha_i\right) - u_i\left(w_i, w_j, t_i^1, \alpha_i\right)$$

is increasing.

The term "increasing differences" is helpful for showing that the optimal transfer of person i (weakly) increases when the intensity of his altruism increases. Because the transfer is selected from a half-open interval, we cannot directly use a result obtained by Topkis (1978). By formulating an extended version of Topkis' Monotonicity Theorem, the following lemma resolves this difficulty. (A formulation and a proof of this version of the theorem are in Appendix B.)

Lemma 1. The optimal transfer $t_i^*(\alpha_i)$ is an increasing and continuous function of α_i.

Proof. The proof is in Appendix A.

The next lemma establishes that for any given altruistic transfer, it is possible to find an intensity of altruism for which the transfer is optimal. As was already mentioned, this lemma will be useful in showing in Claim 1 that Assumption 2 constitutes not only a sufficient condition but also a necessary condition for the absolute risk aversion of altruistic person i to decrease as the intensity of his altruism increases.

Lemma 2. For any $\hat{t} \in [0, w_i)$ there exists $\hat{\alpha}_i \in (0, 1)$ such that $t_i^*(\hat{\alpha}_i) = \hat{t}$.

[1] In this paper, we refer to "decreasing" and "increasing" in the weak sense: we say that the function $f(\cdot)$ is *decreasing* if when $x < y$, then $f(x) \geq f(y)$. And we say that the function $f(\cdot)$ is *increasing* if when $x < y$, then $f(x) \leq f(y)$.

Proof. The proof is in Appendix A.

Claim 1. Suppose that the functions $v(\cdot)$ and $u_j(\cdot)$ satisfy Assumption 1. Let w_i be the level of wealth of person i, and let w_j be the level of wealth of person j. Let t_i^* be the solution of problem (2). Then the following six results hold.

(i) There is a critical level of the intensity of altruism $\overline{\alpha}_i \in (0, 1)$, defined as

$$\overline{\alpha}_i \equiv \frac{v'(w_i)}{v'(w_i) + u_j'(w_j)}, \tag{4}$$

such that altruistic person i will transfer part of his wealth whenever the intensity of his altruism exceeds $\overline{\alpha}_i$. If the intensity of person i's altruism is at most $\overline{\alpha}_i$, then altruistic person i will not transfer any part of his wealth.

(ii) The absolute risk aversion of altruistic person i satisfies the following condition:

$$ARA_i(\alpha_i) = \begin{cases} \dfrac{1}{B_i(w_i)} & \text{if} \quad \alpha_i \leq \overline{\alpha}_i, \\ \dfrac{1}{B_i[w_i - t_i^*(\alpha_i)] + B_j[w_j + t_i^*(\alpha_i)]} & \text{if} \quad \alpha_i > \overline{\alpha}_i. \end{cases}$$

(iii) The absolute risk aversion of altruistic person i is continuous on $\alpha_i \in (0, \overline{\alpha}_i) \cup (\overline{\alpha}_i, 1)$ and discontinuous at $\alpha_i = \overline{\alpha}_i$ in such a way that the switch is from a higher value to a lower value.

(iv) The absolute risk aversion of altruistic person i is decreasing in α_i if and only if w_i and w_j satisfy Assumption 2.

(v) Suppose that w_i and w_j satisfy Assumption 2 and that altruistic person i engages optimally in a wealth transfer to person j, that is, suppose that the condition $\alpha_i > \overline{\alpha}_i$ holds. If the condition in (3) holds with equality for every $t_i \in (0, w_i)$, then the absolute risk aversion of altruistic person i acquires a constant value. If the condition in (3) holds with strict inequality for every $t_i \in (0, w_i)$, then the absolute risk aversion of altruistic person i is strictly decreasing.

(vi) Suppose that w_i and w_j satisfy Assumption 2. Then, under the condition that altruistic person i engages optimally in a wealth transfer to person j, person i is strictly less risk averse than a comparable person who is not altruistic, meaning a person for whom $\alpha_i = 0$.

Proof. The proof is in Appendix A.

Observation 1. Part (vi) of Claim 1 can be perceived as a generalization of Claim 1 in Stark (2024) to a broader class of utility functions and for a different measure of risk aversion.

It is natural to next inquire whether our approach can enable us to ascertain how the facility of an altruistic transfer influences the absolute risk aversion of the *beneficiary* of the transfer. For a beneficiary of an altruistic transfer who receives the transfer $t_i^*(\alpha_i)$, the utility function takes the form

$$u_j^*(w_i, w_j, \alpha_i) \equiv u_j[w_j + t_i^*(\alpha_i)]. \tag{5}$$

The coefficient of absolute risk aversion of this beneficiary (using $u_j^*(w_j)$ for $u_j^*(w_i, w_j, \alpha_i)$ whenever w_i is held constant) is

$$ARA_j(\alpha_i) = -\frac{\dfrac{\partial^2 u_j^*(w_i, w_j, \alpha_i)}{\partial w_j^2}}{\dfrac{\partial u_j^*(w_i, w_j, \alpha_i)}{\partial w_j}}.$$

We define a fifth auxiliary function

$$G_j(x_i, x_j) \equiv \frac{A_j(x_j)}{A_i(x_i) + A_j(x_j)} = \frac{B_i(x_i)}{B_i(x_i) + B_j(x_j)},$$

where, as already noted, $x_i \in (0, \overline{w})$ and $x_j \in (\underline{w}, \underline{w} + \overline{w})$.

As a preparatory step for stating and proving our second claim, we formulate a lemma about an elementary property of $G_j(\cdot)$. In combination with Assumption 2, this property is needed in order to enable us to show that the beneficiary of an altruistic transfer is less risk averse than a comparable person who is not in receipt of an altruistic transfer.

Lemma 3. Let $x_i \in (0, \overline{w})$ and $x_j \in (\underline{w}, \underline{w} + \overline{w})$ be given such that $B_i'(x_i) \leq B_j'(x_j)$. Then

$$H(x_i, x_j) \equiv -\frac{\partial G_j(x_i, x_j)}{\partial x_j} - \left[\frac{\partial G_j(x_i, x_j)}{\partial x_i} - \frac{\partial G_j(x_i, x_j)}{\partial x_j}\right] G_j(x_i, x_j) \geq 0. \tag{6}$$

Proof. The proof is in Appendix A.

Claim 2. Suppose that $v(\cdot)$ and $u_j(\cdot)$ satisfy Assumption 1, and that the levels of wealth of persons i and j, respectively w_i and w_j, satisfy Assumption 2. Then, under the condition that altruistic person i engages optimally in a wealth transfer to person j, namely under the condition $\alpha_i > \overline{\alpha}_i$, person j is (strictly) less risk averse than had he not been in receipt of a transfer.

Proof. The proof is in Appendix A.

Observation 2. Claim 2 can be perceived as a generalization of part (i) of Claim 2 in Stark et al. (2022) to a broader class of utility functions and for a different measure of risk aversion.

Summarizing this section, we restate our main points of interest. Is a person who is more altruistic less risk averse than a comparable person who is less altruistic? Is a person who is a beneficiary of an altruistic transfer less risk averse than a comparable person who does not receive an altruistic transfer? To address the first point of interest, in Claim 1 we applied an additional condition concerning the class of the utility functions of persons i and j and the wealth levels of these persons. We showed that the conditions concerning the levels of wealth were necessary and sufficient for establishing an inverse relationship between the intensity of altruism and the absolute risk aversion of an altruistic person. In addition, we found that an altruistic person who was an active donor was less risk averse than a comparable person who was not an active altruistic donor. To address the second point of interest, in Claim 2 we showed that under the same conditions as those which underlie Claim 1, the beneficiary of an altruistic transfer is less risk averse than a comparable person who is not in receipt of an altruistic transfer.

3. A special case of the general formulation: The CRRA utility function

In this section, we show that the general model presented in the preceding section can be illustrated by the special case of a constant relative risk aversion (CRRA) utility function. To enable us to draw a complete picture, we begin with Claim 3, where we show that the CRRA utility function delivers a sufficient condition for the absolute risk aversion of an altruistic person who is an active donor to be a constant, and that this person is less risk averse than a comparable nonaltruistic person. Next, in Claim 4, we show how the facility of an altruistic transfer influences the absolute risk aversion of the beneficiary of the transfer: the CRRA utility function delivers a sufficient condition for the absolute risk aversion of a beneficiary of an altruistic transfer to be lower than the absolute risk aversion of a comparable person who is not a beneficiary of an altruistic transfer. Thus, in terms of risk-taking behavior, under CRRA utility functions, as under the general setting studied in Section 2, a "population" endowed with altruism is uniformly more willing to take risks than a comparable "population" devoid of altruism.

Under CRRA, (1) takes the form

$$u_i(w_i, w_j, t_i, \alpha_i) = (1-\alpha_i)\frac{(w_i - t_i)^{1-\eta_i} - 1}{1-\eta_i} + \alpha_i \frac{(w_j + t_i)^{1-\eta_j} - 1}{1-\eta_j}, \tag{7}$$

where $\eta_i \in (0, 1)$ and $\eta_j \in (0, 1)$ are constants. We subsequently assume that $\eta_i = \eta_j = \eta$.

Claim 3. Suppose that the utility of altruistic person i takes the form (7). Then for every $w_i > 0$ and $w_j > 0$, the absolute risk aversion of person i who engages optimally in a wealth transfer to person j is constant for α_i. Moreover, person i is less risk averse than a comparable nonaltruistic person.

Proof. The proof is in Appendix A.

Observation 3. The second part of Claim 3 resonates in a broader context and for a different measure of risk aversion a result reported in Stark (2024) and can be perceived as a corollary of Claim 1.

Claim 4. Suppose that the utility function of altruistic person i is represented by the CRRA utility function (7). Then for every $w_i > 0$ and $w_j > 0$, person j, who is a recipient of an altruistic transfer, is strictly less risk averse than a comparable person who is not in receipt of an altruistic transfer.

Proof. The proof is in Appendix A.

Observation 4. Claim 4 resonates in a broader context and for a different measure of risk aversion a corresponding result reported in Stark et al. (2022) and can be perceived as a corollary of Claim 2.

4. A special case of the general formulation: The logarithmic utility function

Following Bernheim and Stark (1988) and Stark (1999), we consider a logarithmic representation of the utility functions of persons i and j. In such a case

$$u_i(w_i, w_j, t_i, \alpha_i) = (1 - \alpha_i)\ln(w_i - t_i) + \alpha_i \ln(w_j + t_j). \tag{8}$$

That is, the component functions in (1), $v(\cdot)$ and $u_i(\cdot)$, take, respectively, the forms of $v(w_i - t_i) = \ln(w_i - t_i)$, and $u_j(w_j + t_i) = \ln(w_j + t_i)$. It is nice to note that by an application of L'Hôpital's rule with $\eta = 1$, the specification in (8) can be elicited directly from the CRRA utility function (7). When the utility of person i is expressed by (8), we can obtain the same results as those reported in Claims 3 and 4.

Claim 3 says that the absolute risk aversion of an altruistic person i is constant for the set of α_i such that person i optimally engages in a wealth transfer to person j, and that person i, who is an active donor, is less risk averse than a comparable nonaltruistic person. Following steps that are similar to the ones taken in the proof of Claim 3, we obtain $\underline{w} = 0$ and $\overline{w} = w_i + 1$. For every $x_i \in (0, \overline{w})$, we note that $v'(x_i) = 1/x_i$, and $v''(x_i) = -1/x_i^2$. Similarly, for every $x_j \in (\underline{w}, \underline{w} + \overline{w})$, we note that $u_j'(x_j) = 1/x_j$ and $u_j''(x_j) = -1/x_j^2$. Moreover, the expressions $B_i(\cdot)$ and $B_j(\cdot)$ now take, respectively, the forms of $B_i(x_i) = x_i$ and $B_j(x_j) = x_j$. Thus, $B_i'(x_i) = 1$ and $B_j'(x_j) = 1$. Therefore, condition (3) with equality straightforwardly holds in this logarithmic case. Then, as per Claim 1, the absolute risk aversion of person i is constant for the set of intensities for which person i is an active donor. Moreover, an altruistic person who is an active donor is less risk averse than a comparable person who is not altruistic.

Regarding the counterpart of Claim 4, which says that a beneficiary of an altruistic transfer is less risk averse than a comparable person who is not in receipt of an altruistic transfer, we proceed in a similar way to the proof of Claim 4, and we recall that the levels of wealth of both persons satisfy Assumption 2. As a result, from Claim 2 we conclude that person j, who is a beneficiary of an altruistic transfer, is less risk averse than a comparable person who is not in receipt of an altruistic transfer.

In sum: in the case of logarithmic utility functions, altruistic person i is less risk averse than a comparable person who is not altruistic. And person j, who is a recipient of an altruistic transfer, is less risk averse than a comparable person who is not a beneficiary of an altruistic transfer.

Appendix A. Proofs of Lemmas 1-3 and Claims 1-4

Proof of Lemma 1.

To prove this lemma, we draw on Lemma B.2 in Appendix B, in that when a utility function is strictly concave in t_i and has increasing differences in (α_i, t_i), the (unique) optimal value of t_i is increasing in α_i. Given this, we show that $u_i(w_i, w_j, t_i, \alpha_i)$ has increasing differences in (α_i, t_i). Let $t_i^1 \in [0, w_i)$ and $t_i^2 \in [0, w_i)$ be such that $t_i^2 > t_i^1$. Recalling (1) and from Assumption 1(ii) that the functions $v'(\cdot)$ and $u_j'(\cdot)$ are strictly positive, we obtain

$$\frac{\partial^2 u_i(w_i, w_j, t_i, \alpha_i)}{\partial t_i \partial \alpha_i} = v'(w_i - t_i) + u_j'(w_j + t_i) \geq 0.$$

As a consequence of $t_i^2 > t_i^1$, we obtain

$$\int_{t_i^1}^{t_i^2} \frac{\partial^2 u_i(w_i, w_j, t_i, \alpha_i)}{\partial t_i \partial \alpha_i} dt_i = \frac{\partial [u_i(w_i, w_j, t_i^2, \alpha_i) - u_i(w_i, w_j, t_i^1, \alpha_i)]}{\partial \alpha_i} \geq 0.$$

We then conclude that the function $u_i(w_i, w_j, t_i, \alpha_i)$ has increasing differences in (α_i, t_i). From Assumption 1(i) that the functions $v(\cdot)$ and $u_j(\cdot)$ are strictly concave, we infer that $v(w_i - t_i)$ and $u_j(w_j + t_i)$ are strictly concave in t_i. Then the function $u_i(w_i, w_j, t_i, \alpha_i)$ in (1) is strictly concave in t_i, and the optimal transfer $t_i^*(\alpha_i)$ is uniquely determined. Therefore, by Lemma B.2, $t_i^*(\cdot)$ is an increasing and continuous function of α_i. Q.E.D.

Proof of Lemma 2.

We select any $\hat{t} \in [0, w_i)$. We define $\hat{\alpha}_i \equiv v'(w_i - \hat{t})/[v'(w_i - \hat{t}) + u_j'(w_j + \hat{t})]$. For this intensity of altruism, the utility function of altruistic person i takes the form $u_i(w_i, w_j, t, \hat{\alpha}_i) = (1 - \hat{\alpha}_i)v(w_i - t) + \hat{\alpha}_i u_j(w_j + t)$. Then

$$\frac{\partial u_i}{\partial t_i}(w_i, w_j, \hat{t}, \hat{\alpha}_i) = -(1 - \hat{\alpha}_i)v'(w_i - \hat{t}) + \hat{\alpha}_i u_j'(w_j + \hat{t}) = 0.$$

Recalling again that $v(w_i - t_i)$ and $u_j(w_j + t_i)$ are strictly concave in t_i, we infer (in view of (1) with α_i replaced by $\hat{\alpha}_i$) that the function $u_i(w_i, w_j, t_i, \hat{\alpha}_i)$ is strictly concave in t_i. Thus, $t_i^*(\hat{\alpha}_i) = \hat{t}$. Q.E.D.

Proof of Claim 1.

We first attend to part (i). We need to show that the intensity $\bar{\alpha}_i$ in (4) satisfies the following condition: $t_i^*(\alpha_i) = 0$ for $\alpha_i \leq \bar{\alpha}_i$, and $t_i^*(\alpha_i) > 0$ for $\alpha_i > \bar{\alpha}_i$. Clearly, from (4) it follows that $\bar{\alpha}_i \in (0, 1)$. We show that $t_i^*(\alpha_i) = 0$ for any $\alpha_i \leq \bar{\alpha}_i$. For such α_i, it follows from (4) that

$$\frac{\partial u_i(w_i, w_j, 0, \alpha_i)}{\partial t_i} = -(1 - \alpha_i)v'(w_i) + \alpha_i u_j'(w_j) \leq 0. \tag{9}$$

We recall once again that the function $u_i(w_i, w_j, t_i, \alpha_i)$ is strictly concave in t_i. Drawing on (9), we have proven that $t_i^*(\alpha_i) = 0$. In a similar manner, we can show that for any $\alpha_i > \overline{\alpha}_i$, in the inequality in (9) the sign changes from "$\leq$" to "$>$". Therefore, $\partial u_i(w_i, w_j, 0, \alpha_i)/\partial t_i > 0$, and $t_i^*(\alpha_i) > 0$.

We next attend to part (ii). From part (i) we know that $ARA_i(\alpha_i) = 1/B_i(w_i)$ whenever $\alpha_i \leq \overline{\alpha}_i$. Let $\alpha_i > \overline{\alpha}_i$. From Assumption 1, that the functions $v(\cdot)$ and $u_j(\cdot)$ have continuous derivatives on $(0, \overline{w})$ and on $(\underline{w}, \underline{w} + \overline{w})$ respectively, we infer that

$$\frac{\partial u_i(w_i, w_j, 0, \alpha_i)}{\partial t_i} = -(1-\alpha_i)v'(w_i) + \alpha_i u_j'(w_j).$$

From here and from (4), we obtain that $\partial u_i(w_i, w_j, 0, \alpha_i)/\partial t_i > 0$. From Assumption 1 that $v(\cdot)$ has an infinite right-hand side derivative at zero, we obtain in turn that $\lim_{t_i \to w_i^-} \partial u_i(w_i, w_j, t_i, \alpha_i)/\partial t_i = -\infty$. Given this, and that the function $u_i(w_i, w_j, t_i, \alpha_i)$ is strictly concave in t_i, we infer that the optimal transfer $t_i^*(\cdot)$ is strictly positive and that it satisfies

$$\frac{\partial u_i(w, w, t_i^*(\alpha_i), \alpha_i)}{\partial t_i} = -(1-\alpha_i)v'[w_i - t_i^*(\alpha_i)] + \alpha_i u_j'[w_j + t_i^*(\alpha_i)] = 0. \quad (10)$$

Therefore, by the Implicit Function Theorem applied to

$$F(w_i, w_j, t_i) \equiv -(1-\alpha_i)v'(w_i - t_i) + \alpha_i u_j'(w_j + t_i) \quad (11)$$

- while we hold w_j constant - we find that $t_i^*(w_i)$ satisfies the condition

$$\frac{dt_i^*(w_i)}{dw_i} = -\frac{\dfrac{\partial F(w_i, w_j, t_i^*(w_i))}{\partial w_i}}{\dfrac{\partial F(w_i, w_j, t_i^*(w_i))}{\partial t_i}} = \frac{(1-\alpha_i)v''(w_i - t_i^*(w_i))}{(1-\alpha_i)v''(w_i - t_i^*(w_i)) + \alpha_i u_j''(w_j + t_i^*(w_i))}. \quad (12)$$

In deriving (12) we denote the optimal transfer by $t_i^*(w_i)$, where we consider w_i as a variable and where we hold α_i constant.

The Envelope Theorem applied to the function $u_i(w_i, w_j, t_i, \alpha_i)$, where we hold w_j and α_i constant, yields

$$\frac{\partial u_i^*(w_i, w_j, \alpha_i)}{\partial w_i} = \frac{\partial u_i(w_i, w_j, t_i^*(w_i), \alpha_i)}{\partial w_i} = (1-\alpha_i)v'(w_i - t_i^*(w_i)). \quad (13)$$

Differentiating the two sides in (13) with respect to w_i and then replacing $\frac{dt_i^*(w_i)}{dw_i}$ with the third term in (12), we obtain

$$\frac{\partial^2 u_i^*(w_i, w_j, \alpha_i)}{\partial w_i^2} = (1-\alpha_i)v''(w_i - t_i^*(w_i))\left[1 - \frac{dt_i^*(w_i)}{dw_i}\right]$$
$$= \frac{(1-\alpha_i)\alpha_i v''(w_i - t_i^*(w_i))u_j''(w_j + t_i^*(w_i))}{(1-\alpha_i)v''(w_i - t_i^*(w_i)) + \alpha_i u_j''(w_j + t_i^*(w_i))}. \quad (14)$$

As a result of dividing $u_j''(w_j + t_i^*(w_i))$ by $v'(w_i - t_i^*(w_i))$ and rewriting $v'(w_i - t_i^*(w_i))$ on the basis of the second equality in (10), we obtain

$$\frac{u_j''(w_j + t_i^*(w_i))}{v'(w_i - t_i^*(w_i))} = \frac{(1-\alpha_i)}{\alpha_i} \frac{u_j''(w_j + t_i^*(w_i))}{u_j'(w_j + t_i^*(w_i))}.$$

Given this, we divide the first term and the last term in (14) by $\partial u_i^*(w_i, w_j, \alpha_i)/\partial w_i$, and we then replace this derivative with the third term in (13). We thus obtain

$$\frac{\dfrac{\partial^2 u_i^*(w_i, w_j, \alpha_i)}{\partial w_i^2}}{\dfrac{\partial u_i^*(w_i, w_j, \alpha_i)}{\partial w_i}} = \frac{\alpha_i \dfrac{v''(w_i - t_i^*(w_i))}{v'(w_i - t_i^*(w_i))} \dfrac{u_j''(w_j + t_i^*(w_i))}{v'(w_i - t_i^*(w_i))}}{(1-\alpha_i)\dfrac{v''(w_i - t_i^*(w_i))}{v'(w_i - t_i^*(w_i))} + \alpha_i \dfrac{u_j''(w_j + t_i^*(w_i))}{v'(w_i - t_i^*(w_i))}}$$

$$= \frac{\dfrac{v''(w_i - t_i^*(w_i))}{v'(w_i - t_i^*(w_i))} \dfrac{u_j''(w_j + t_i^*(w_i))}{u_j'(w_j + t_i^*(w_i))}}{\dfrac{v''(w_i - t_i^*(w_i))}{v'(w_i - t_i^*(w_i))} + \dfrac{u_j''(w_j + t_i^*(w_i))}{u_j'(w_j + t_i^*(w_i))}}.$$

The preceding expression and the definitions of auxiliary functions $A_i(\cdot)$ and $A_j(\cdot)$ imply that

$$ARA_i(\alpha_i) = \frac{A_i(w_i - t_i^*(w_i))A_j(w_j + t_i^*(w_i))}{A_i(w_i - t_i^*(w_i)) + A_j(w_j + t_i^*(w_i))}.$$

We equivalently express $ARA_i(\alpha_i)$ - while holding w_i constant we again denote the optimal transfer by $t_i^*(\alpha_i)$ - as

$$ARA_i(\alpha_i) = \frac{1}{B_i[w_i - t_i^*(\alpha_i)] + B_j[w_j + t_i^*(\alpha_i)]}. \quad (15)$$

We next attend to part (iii). Recalling part (ii), we find that $ARA_i(\alpha_i)$ is a constant equal to $1/B_i(w_i)$ on $\alpha_i \in (0, \overline{\alpha}_i]$. In addition, recalling from Assumption 1 that $v''(\cdot)$ and $u_j''(\cdot)$ are continuous on their domains, as well as from Lemma 1 that $t_i^*(\alpha_i)$ is a continuous function, we infer that $ARA_i(\alpha_i)$ is continuous on $\alpha_i \in (\overline{\alpha}_i, 1)$. To complete the proof of part (iii), what remains to be shown is that $ARA_i(\alpha_i)$ switches from a higher value to a lower value. We take the limit in (15) whenever $\alpha_i > \overline{\alpha}_i$ and α_i tends to $\overline{\alpha}_i$ from the right-hand side. Drawing on Lemma 1, that $t_i^*(\cdot)$ is continuous, it follows that $t_i^*(\alpha_i)$ tends to $t_i^*(\overline{\alpha}_i) = 0$. Applying (15) and recalling that $v''(\cdot)$ and $u_j''(\cdot)$ are continuous on their domains, we obtain

$$\lim_{\alpha_i \to \overline{\alpha}_i^+} ARA_i(\alpha_i) = \frac{1}{B_i(w_i) + B_j(w_j)} < \frac{1}{B_i(w_i)} = ARA_i(\overline{\alpha}_i).$$

In Figure 1, we plot $ARA_i(\alpha_i)$ against α_i, thereby illustrating the switch alluded to earlier.

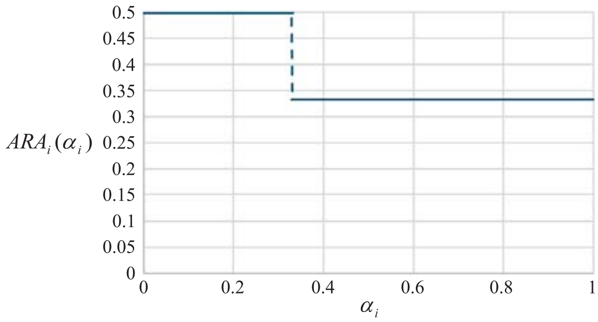

Figure 1. An example of a switch of $ARA_i(\alpha_i)$ at $\overline{\alpha}_i$: the horizontal axis measures the intensity of altruism, and the vertical axis measures the level of absolute risk aversion. The dashed line indicates the extent of the switch at $\overline{\alpha}_i$ from a higher level of absolute risk aversion to a lower level of absolute risk aversion. For drawing the Figure, we use the functions $v(w_i - t_i) = \ln(w_i - t_i)$ and $u_j(w_j + t_i) = \ln(w_j + t_i)$ and the levels of wealth $w_i = 2$ and $w_j = 1$.

We next attend to part (iv). We consider the relationship between the optimal transfer and the intensity of altruism. Given part (iii), we only need to analyze the absolute risk aversion of person i with respect to the intensity of his altruism, provided the optimal altruistic transfer is strictly positive. To this end, suppose that $\alpha_i > \overline{\alpha}_i$. For any $t_i \in (0, w_i)$ define $\psi(t_i) \equiv 1/[B_i(w_i - t_i) + B_j(w_j + t_i)]$. Then the first derivative of $\psi(t_i)$ is

$$\psi'(t_i) = \frac{B_i'(w_i - t_i) - B_j'(w_j + t_i)}{[B_i(w_i - t_i) + B_j(w_j + t_i)]^2}. \tag{16}$$

Suppose that the levels of wealth of persons i and j satisfy Assumption 2. Then $\psi'(t_i) \le 0$, and hence $\psi(\cdot)$ is decreasing. Applying Lemma 1, we know that $t_i^*(\alpha_i)$ is increasing. Drawing on (15), it follows that $ARA_i(\alpha_i) = \psi[t_i^*(\alpha_i)]$. Hence, $ARA_i(\alpha_i)$ is decreasing in α_i. Next, by contradiction, we prove that for the levels of wealth of both persons, Assumption 2 is satisfied whenever $ARA_i(\alpha_i)$ is decreasing in α_i. Suppose then that at some $\hat{t}_i \in (0, w_i)$, (3) is violated, that is, that

$$B_i'(w_i - \hat{t}_i) - B_j'(w_j + \hat{t}_i) > 0.$$

From Lemma 2 we know that the range of $t_i^*(\alpha_i)$ is the interval $[0, w_i)$. Therefore, there exists $\hat{\alpha}_i \in (\overline{\alpha}_i, 1)$ such that $\hat{t}_i = t_i^*(\hat{\alpha}_i)$. By Assumption 1, we find that $B_i'(\cdot)$ and $B_j'(\cdot)$ are both continuous on their domains. From Lemma 1 we already know that $t_i^*(\alpha_i)$ is continuous. Then there is an open interval $I \subset (\overline{\alpha}_i, 1)$ containing $\hat{\alpha}_i$ such that for any $\alpha_i \in I$ the following holds:

$$B_i'[w_i - t_i^*(\alpha_i)] - B_j'[w_j + t_i^*(\alpha_i)] > 0.$$

From the definition of $\psi(\cdot)$ and in view of (16), we obtain that $\psi(\cdot)$ is strictly increasing over the range of $t_i^*(\alpha_i)$ for $\alpha_i \in I$. Applying once again (15), that $ARA_i(\alpha_i) = \psi[t_i^*(\alpha_i)]$, we infer that $ARA_i(\alpha_i)$ is increasing for $\alpha_i \in I$, which leads us to conclude that $ARA_i(\alpha_i)$ is strictly increasing on $\alpha_i \in I$. Indeed, if this were not the case, then $ARA_i(\alpha_i)$ would be a constant-valued function on I. Because $\psi(\cdot)$

is strictly increasing, $t_i^*(\alpha_i)$ has to be a constant-valued function on I equal to $\hat{t}_i$. Then the equation $v'(w_i - \hat{t}_i) = u_j'(w_j + \hat{t}_i) = 0$ would hold. From Assumption 1, that $v'(\cdot)$ and $u_j'(\cdot)$ are strictly positive on their domains, we conclude that this is impossible. Therefore, $ARA_i(\alpha_i)$ is strictly increasing on I. This protocol of contradiction completes the proof of part (iv).

We next attend to part (v). If the condition in (3) holds with equality, then the function $B_i(w_i - t_i) + B_j(w_j + t_i)$ is constant for t_i. Then $\psi(t_i)$ is constant. Applying once again (15), that $ARA_i(\alpha_i) = \psi[t_i^*(\alpha_i)]$, we infer that $ARA_i(\alpha_i)$ is constant for $\alpha_i > \overline{\alpha}_i$. If the condition in (3) holds with strict inequality, then $\psi(t_i)$ is strictly decreasing. As per part (iv), we conclude that $t_i^*(\alpha_i)$ cannot be constant on any nondegenerate subinterval of $(\overline{\alpha}_i, 1)$. Thus, $t_i^*(\alpha_i)$ is strictly increasing and, hence, $ARA_i(\alpha_i)$ is strictly decreasing on $\alpha_i > \overline{\alpha}_i$.

We finally attend to part (vi). Applying part (i) we obtain that $t_i^*(\alpha_i) = 0$ whenever $\alpha_i \leq \overline{\alpha}_i$ and that $t_i^*(\alpha_i) > 0$ whenever $\alpha_i > \overline{\alpha}_i$. Next, from part (ii) we know that $ARA_i(\alpha_i) = 1/B_i(w_i)$ whenever $t_i^*(\alpha_i) = 0$. In part (iii), we established that $ARA_i(\alpha_i)$ was discontinuous at $\overline{\alpha}_i$ and that it switched from a higher value to a lower value. Because w_i and w_j satisfy Assumption 2, by part (iv) we infer that $ARA_i(\alpha_i)$ is decreasing. Therefore, if $t_i^*(\alpha_i) > 0$, then person i is (strictly) less risk averse than a comparable person who is not altruistic. Q.E.D.

Proof of Lemma 3.

For proving the inequality in (6), we only need to show that $H(x_i, x_j) \geq 0$ whenever $B_i'(x_i) \leq B_j'(x_j)$. From the definition of the auxiliary function $G_j(\cdot)$ we obtain

$$-\frac{\partial G_j(x_i, x_j)}{\partial x_j} = \frac{B_i(x_i)B_j'(x_j)}{[B_i(x_i) + B_j(x_j)]^2}, \tag{17}$$

and

$$\frac{\partial G_j(x_i, x_j)}{\partial x_i} = \frac{B_j(x_j)B_i'(x_i)}{[B_i(x_i) + B_j(x_j)]^2}. \tag{18}$$

Inserting the right-hand sides of (17) and (18) into $H(\cdot)$, we obtain

$$\begin{aligned} H(x_i, x_j) &= \frac{B_i(x_i)B_j'(x_j)}{[B_i(x_i) + B_j(x_j)]^2} \\ &\quad - \left[\frac{B_i(x_i)B_j'(x_j)}{[B_i(x_i) + B_j(x_j)]^2} + \frac{B_j(x_j)B_i'(x_i)}{[B_i(x_i) + B_j(x_j)]^2}\right]\frac{B_i(x_i)}{[B_i(x_i) + B_j(x_j)]} \\ &\geq \frac{B_i(x_i)B_j'(x_j)}{[B_i(x_i) + B_j(x_j)]^2} - \frac{B_i(x_i) + B_j(x_j)}{[B_i(x_i) + B_j(x_j)]^2}\frac{B_i(x_i)B_j'(x_j)}{[B_i(x_i) + B_j(x_j)]} = 0, \end{aligned}$$

where the inequality draws on $B_i'(x_i) \leq B_j'(x_j)$. Thus, $H(x_i, x_j) \geq 0$, and the inequality in (6) is satisfied. Q.E.D.

Proof of Claim 2.

Let $\overline{ARA}_j$ be the coefficient of absolute risk aversion of person j who is not a beneficiary of an altruistic transfer so that $t_i^*(\alpha_i) = 0$. We have

$$\overline{ARA}_j = -u_j''(w_j)/u_j'(w_j) = 1/B_j(w_j). \tag{19}$$

Next, we formulate the coefficient of absolute risk aversion of person j who receives an altruistic transfer. That is, $t_i^*(\alpha_i) > 0$. Recalling once again that $u_i(\cdot)$ is strictly concave in t_i, we see that the optimal altruistic transfer satisfies the second equality in (10). Equivalently,

$$u_j'[w_j + t_i^*(\alpha_i)] = \frac{(1-\alpha_i)}{\alpha_i} v'[w_i - t_i^*(\alpha_i)]. \tag{20}$$

Thus, from the Implicit Function Theorem applied to $F(w_i, w_j, t_i)$ - recalling (11), considering w_j as a variable, and holding w_i constant - we obtain that

$$\begin{aligned}
\frac{dt_i^*(w_j)}{dw_j} &= -\frac{\dfrac{\partial F(w_i, w_j, t_i^*(w_j))}{\partial w_j}}{\dfrac{\partial F(w_i, w_j, t_i^*(w_j))}{\partial t_i}} \\
&= -\frac{\alpha_i u_j''(w_j + t_i^*(w_j))}{(1-\alpha_i) v''(w_i - t_i^*(w_j)) + \alpha_i u_j''(w_j + t_i^*(w_j))} \\
&= -\frac{\alpha_i \dfrac{u_j''(w_j + t_i^*(w_j))}{u_j'(w_j + t_i^*(w_j))}}{(1-\alpha_i)\dfrac{v''(w_i - t_i^*(w_j))}{u_j'(w_j + t_i^*(w_j))} + \alpha_i \dfrac{u_j''(w_j + t_i^*(w_j))}{u_j'(w_j + t_i^*(w_j))}}.
\end{aligned} \tag{21}$$

Replacing $u_j'(w_j + t_i^*(w_j))$ in the denominator of the term in the last line in (21) with the right-hand side term in (20) we obtain

$$\begin{aligned}
\frac{dt_i^*(w_j)}{dw_j} &= -\frac{\dfrac{u_j''(w_j + t_i^*(w_j))}{u_j'(w_j + t_i^*(w_j))}}{\dfrac{v''(w_i - t_i^*(w_j))}{v'(w_i - t_i^*(w_j))} + \dfrac{u_j''(w_j + t_i^*(w_j))}{u_j'(w_j + t_i^*(w_j))}} \\
&= -\frac{A_j(w_j + t_i^*(w_j))}{A_i(w_i - t_i^*(w_j)) + A_j(w_j + t_i^*(w_j))} \\
&= -G_j(w_i - t_i^*(w_j), w_j + t_i^*(w_j)).
\end{aligned} \tag{22}$$

From Assumption 1 that $v(\cdot)$ and $u_j(\cdot)$ are three times continuously differentiable on $(0, \overline{w})$ and on $(\underline{w}, \underline{w} + \overline{w})$, we find that the following expression holds:

$$\frac{d^2 t_i^*(w_j)}{dw_j^2} = -\frac{\partial G_j(w_i - t_i^*(w_j), w_j + t_i^*(w_j))}{\partial x_j} + \tilde{G}(w_i - t_i^*(w_j), w_j + t_i^*(w_j)) \frac{dt_i^*(w_j)}{dw_j}$$
$$= -\frac{\partial G_j(x_i^*, x_j^*)}{\partial x_j} - \left[\frac{\partial G_j(x_i^*, x_j^*)}{\partial x_i} - \frac{\partial G_j(x_i^*, x_j^*)}{\partial x_j}\right] G_j(x_i^*, x_j^*) \geq 0, \tag{23}$$

where

$$\tilde{G}(w_i - t_i^*(w_j), w_j + t_i^*(w_j)) \equiv \frac{\partial G_j(w_i - t_i^*(w_j), w_j + t_i^*(w_j))}{\partial x_i} - \frac{\partial G_j(w_i - t_i^*(w_j), w_j + t_i^*(w_j))}{\partial x_j},$$

$x_i^* \equiv w_i - t_i^*(w_j)$ and $x_j^* \equiv w_j + t_i^*(w_j)$, and the inequality in the second line of (23) follows (in view of (3) in Assumption 2) from Lemma 3.

Next, we determine ARA_j. The utility function, $u_j^*(\cdot)$, of person j is already displayed in (5). Differentiating $u_j^*(\cdot)$ once and twice, we obtain, respectively, that

$$[u_j^*(w_j)]' = u_j'(w_j + t_i^*(w_j))\left[1 + \frac{\partial t_i^*(w_j)}{\partial w_j}\right],$$

and

$$[u_j^*(w_j)]'' = u_j''(w_j + t_i^*(w_j))\left[1 + \frac{\partial t_i^*(w_j)}{\partial w_j}\right]^2 + u_j'(w_j + t_i^*(w_j))\frac{\partial^2 t_i^*(w_j)}{\partial w_j^2}.$$

Thus, the absolute risk aversion of person j is

$$ARA_j = -\frac{u_j''(w_j + t_i^*(w_j))}{u_j'(w_j + t_i^*(w_j))}\left[1 + \frac{\partial t_i^*(w_j)}{\partial w_j}\right] - \frac{\dfrac{\partial^2 t_i^*(w_j)}{\partial w_j^2}}{1 + \dfrac{\partial t_i^*(w_j)}{\partial w_j}}.$$

Inserting the second line in (22) into the preceding expression, and recalling the definitions of $B_i(\cdot)$ and $B_j(\cdot)$, we obtain that

$$ARA_j = \frac{1}{B_i(w_i - t_i^*(w_j)) + B_j(w_j + t_i^*(w_j))} - \frac{[B_i(w_i - t_i^*(w_j)) + B_j(w_j + t_i^*(w_j))]\dfrac{\partial^2 t_i^*(w_j)}{\partial w_j^2}}{B_j(w_j + t_i^*(w_j))}.$$

From (23), that $\partial^2 t_i^*(w_j)/\partial w_j^2 \geq 0$, we obtain

$$ARA_j \leq \frac{1}{B_i(w_i - t_i^*(w_j)) + B_j(w_j + t_i^*(w_j))}. \tag{24}$$

Drawing on (3), we find that the function $B_i(w_i - t_i) + B_j(w_j + t_i)$ is increasing in t_i. Thus,

$$B_i(w_i - t_i^*(w_j)) + B_j(w_j + t_i^*(w_j)) \geq B_i(w_i) + B_j(w_j) > B_j(w_j).$$

Combining this inequality with (19) and (24), we conclude that

$$ARA_j \leq \frac{1}{B_i(w_i - t_i^*(w_j)) + B_j(w_j + t_i^*(w_j))} \leq \frac{1}{B_i(w_i) + B_j(w_j)} < \frac{1}{B_j(w_j)} = \overline{ARA}_j.$$

Hence, person j who is not in receipt of an altruistic transfer is more risk averse than a comparable person who receives an altruistic transfer. Q.E.D.

Proof of Claim 3.

We prove the claim by a direct application of Claim 1. We begin with the observation that the utility function (7) is a special case of the utility function (1), where the role of $v(w_i - t_i)$ is fulfilled by $\frac{(w_i - t_i)^{1-\eta} - 1}{1-\eta}$, and the role of $u_j(w_j + t_i)$ is fulfilled by $\frac{(w_j + t_i)^{1-\eta} - 1}{1-\eta}$, with $t_i \in [0, w_i)$. By letting $v(\cdot)$ and $u_j(\cdot)$ assume these forms, we show that they satisfy Assumption 1. We pick $\underline{w} \equiv 0$ and $\overline{w} \equiv w_i + 1$. We obtain straightforwardly that $v'(x_i) = x_i^{-\eta} > 0$ for $x_i \in (0, \overline{w})$, and that $u_j'(x_j) = x_j^{-\eta} > 0$ for $x_j \in (\underline{w}, \underline{w} + \overline{w})$. Differentiating $v'(\cdot)$ and $u_j'(\cdot)$, we find that $v''(x_i) = -\eta x_i^{-\eta-1} < 0$, and that $u_j''(x_j) = -\eta x_j^{-\eta-1} < 0$. Because of this, the functions $v(\cdot)$ and $u_j(\cdot)$ satisfy Assumption 1 and, in particular, they are strictly concave. To round up the proof, we need to verify that Assumption 2 holds, namely that for w_i and for w_j, the inequality in (3) holds for $t_i \in [0, w_i)$. In our current setting:

$$B_i(x_i) = -\frac{v'(x_i)}{v''(x_i)} = \frac{1}{\eta} x_i, \text{ and } B_j(x_j) = -\frac{u_j'(x_j)}{u_j''(x_j)} = \frac{1}{\eta} x_j.$$

Because x_i and x_j are selected arbitrarily (within their permitted ranges), we note that $B_i'(\cdot)$ and $B_j'(\cdot)$ are constant functions on their domains with, respectively, $B_i'(\cdot) = 1/\eta$ and $B_j'(\cdot) = 1/\eta$. In particular, by inserting $x_i = w_i - t_i$ into $B_i'(x_i)$, and $x_j = w_j + t_i$ into $B_j'(x_j)$ (for an arbitrary $t_i \in [0, w_i)$), we obtain that $B_i'(w_i - t_i) = B_j'(w_j + t_i) = 1/\eta$. Thus, w_i and w_j satisfy Assumption 2 because the relation in (3) holds with equality. Therefore, by Claim 1(v), the absolute risk aversion of altruistic person i is constant. And by Claim 1(vi), person i who engages optimally in a wealth transfer to person j is less risk averse than a nonaltruistic person. Q.E.D.

Proof of Claim 4.

We prove this claim by a direct application of Claim 2. The following steps are the same as the steps taken in the proof of Claim 3. In that proof, we established that the CRRA utility function (7) is a special case of the utility function (1), and that the component functions of (7) satisfy Assumption 1. Moreover, the levels of wealth of persons i and j satisfy Assumption 2. Drawing on Claim 2, we infer that the beneficiary of an altruistic transfer is less risk averse than a comparable person who is not in receipt of an altruistic transfer. Q.E.D.

Appendix B. Supplementary results

We present several auxiliary and elementary results.

Lemma B.1. Let $w > 0$, and let $f : [0, w) \rightarrow R$ be a differentiable and strictly concave function such that $\lim_{t \to w^-} f'(t) = -\infty$. Then there is a unique $t^* \in [0, w)$ such that that $f(t^*) = \max_{t \in [0,w)} f(t)$.

Proof. Because $f(\cdot)$ is strictly concave, its derivative is strictly decreasing and continuous. If $f'(0) \leq 0$, then $t^* = 0$ is the unique value that maximizes the function. If $f'(0) > 0$, then the graph of $f'(\cdot)$ intersects the abscissa at exactly one point, t^*, which is the unique value that maximizes the function. Q.E.D.

The next lemma extends Topkis' (1978) Monotonicity Theorem.

Lemma B.2. Let $w > 0$, and let $f : (0, 1) \times [0, w) \rightarrow R$ be a continuous function such that $f(\alpha, \cdot)$ is strictly concave for any $\alpha \in (0, 1)$, and it has increasing differences in (α, t). Then, for any $\alpha \in (0, 1)$, there exists a unique $t^*(\alpha) \in [0, w)$ such that $f(\alpha, t^*(\alpha)) = \max_{t \in [0,w)} f(\alpha, t)$. Moreover, $t^*(\alpha)$ is increasing and continuous.

Proof. Let $\alpha^1 < \alpha^2$. From Lemma B.1 it follows that there exist unique $t^*(\alpha^1)$ and $t^*(\alpha^2)$, both belonging to $[0, w)$, such that $t^*(\alpha^1)$ maximizes $f(\alpha^1, \cdot)$, and that $t^*(\alpha^2)$ maximizes $f(\alpha^2, \cdot)$. Let $w^* = \max[t^*(\alpha^1), t^*(\alpha^2)]$. Because $f(\alpha^1, \cdot)$ and $f(\alpha^2, \cdot)$ are strictly concave, $t^*(\alpha^1)$ maximizes $f(\alpha^1, \cdot)$, and $t^*(\alpha^2)$ maximizes $f(\alpha^2, \cdot)$ on $[0, w^*]$. By applying Topkis' Monotonicity Theorem to the maximization problem of $f(\alpha^1, \cdot)$ such that $t \in [0, w^*]$, and to the maximization problem of $f(\alpha^2, \cdot)$ such that $t \in [0, w^*]$, we obtain $t^*(\alpha^1) \leq t^*(\alpha^2)$. We next establish the continuity of $t^*(\cdot)$. Let $\tilde{\alpha} \in (0, 1)$ be given, and suppose that $I \subset (0, 1)$ is an interval containing $\tilde{\alpha}$ such that $\sup(I) < 1$. From the preceding step it follows that $\tilde{w} \equiv \sup\{t^*(\alpha) : \alpha \in I\} \leq t^*[\sup(I)] < w$. For any $\alpha \in I$ we consider the maximization problem of $f(\alpha, t)$ such that $t \in [0, \tilde{w}]$. Recalling again that for any $\alpha \in I$ $f(\alpha, \cdot)$ is strictly concave, we infer that $t^*(\alpha)$ is the unique value that maximizes $f(\alpha, \cdot)$ such that $t \in [0, \tilde{w}]$. Applying in turn Topkis' Monotonicity Theorem to this problem, we infer that $t^*(\alpha)$ is continuous. Q.E.D.

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